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IDENTIFICATION OF TWO-DIMENSIONAL HEAT FLOWS IN ANISOTROPIC BODIES OF COMPLEX FORM

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A method is proposed for numerical determination of two-dimensional temperature fields in anisotropic bodies with an arbitrary boundary for use in coefficient inverse heat-conduction problems.

In solving coefficient inverse heat-conduction problems, it is necessary to first evaluate the temperature field on the basis of approximate thermophysical characteristics (ATC). The availability of suitable methods and application packages makes it possible, by varying the approximately assigned ATC, to establish the empirical temperature fields that will be used in determining the sought ATC.

Here, we examine the formulation the numerical solution of two-dimensional nonlinear problems of heat conduction in anisotropic bodies in which complex heat transfer is taking place. Without simplications., the method makes it possible to identify full-scale temperature fields that are then used to determine the principal components λ_ξ, λ_η of the thermal conductivity tensor.

The mathematical model has the following form (Fig. 1):

$$c(T)\rho\frac{\partial T}{\partial\tau} = \operatorname{div}(\Lambda \operatorname{grad} T); \tag{1}$$

$$\left(\frac{\alpha}{c_p}\right)_{w1}(I_{e1} - I_{w1}) - \Lambda \operatorname{grad} T/_{w1} - \varepsilon_{w1}\sigma T_{w1}^4 = 0; \tag{2}$$

$$\alpha_{w2}(T_{e2} - T_{w2}) + \Lambda \operatorname{grad} T/_{w2} - \varepsilon_{w2}\sigma T_{w2}^4 = 0; \tag{3}$$

$$T(r,\ 0,\ \tau) = T_{w3}(r,\ \tau);\ T(r,\ \pi,\ \tau) = T_{w4}(r,\ \tau); \tag{4}$$

$$T(r,\ \theta,\ 0) = \varphi(r,\ \theta). \tag{5}$$

In curvilinear coordinates, the components of the thermal conductivity tensor have the form

$$\begin{aligned}\lambda_{rr} &= \lambda_\xi(T)\cos^2(\theta^\nu - \psi) + \lambda_\eta(T)\sin^2(\theta^\nu - \psi);\\ \lambda_{\theta\theta} &= \lambda_\xi(T)\sin^2(\theta^\nu - \psi) + \lambda_\eta(T)\cos^2(\theta^\nu - \psi);\\ \lambda_{r\theta} &= \lambda_{\theta r} = [\lambda_\eta(T) - \lambda_\xi(T)]\sin(\theta^\nu - \psi)\cos(\theta^\nu - \psi),\end{aligned} \tag{6}$$

where $\nu = 1$ for curvilinear coordinates and $\nu = 0$ for cartesian coordinates.

In solving boundary-value problem (1)-(5), we encounter the problem of allowing for the oblique derivative at the boundary wl in boundary condition (2) and its relationship with the behavior of the boundary $r_{w1} = f(\theta)$.

After projecting the balance (2) in the direction of a normal to the boundary wl, the author of [1] obtained the following representation of the heat flux normal to the boundary wl:

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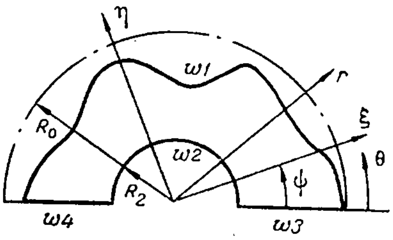

Fig. 1

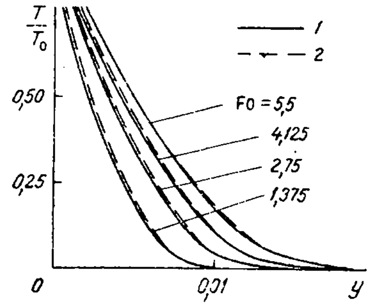

Fig. 2

Fig. 1. Theoretical region.

Fig. 2. Comparison of the numerical solution with the analytical solution from [4]: 1) analytical solution; 2) numerical solution. y, m.

$$\lambda_n \frac{\partial T}{\partial n} = \left[\lambda_{rr} + 2\lambda_{r\theta} \frac{df}{\overset{v}{r}_{w1} d\theta} + \lambda_{\theta\theta} \left(\frac{df}{\overset{v}{r}_{w1} d\theta} \right)^2 \right] \times \left(\frac{\partial T}{\partial r} + \frac{\partial T}{\overset{v}{r}_{w1} \partial\theta} \frac{df}{\overset{v}{r}_{w1} d\theta} \right) \left[1 + \left(\frac{df}{\overset{v}{r}_{w1} d\theta} \right)^2 \right]^{-3/2}. \quad (7)$$

Thus, first of all, this form accounts for the behavior of the boundary wl by means of the function $r_{wl} = f(\theta)$, thus significantly reducing the losses of heat flows on the boundary wl in the numerical realization; secondly, Eq. (7) can be adapted for branching in the coordinate directions r and θ in economical numerical branching methods; third, Eq. (7) includes components of thermal conductivity tensor (6) which are dependent on the principal components λ_ξ and λ_η, rather than on the values of thermal conductivity λ_n in the direction normal to the boundary wl, which is unknown.

The most economical and at the same time accurate method of solving problems of the type (1)-(5) is the method of longitudinal-transverse directions [2]. However, it can be shown [3] that this method is conditionally stable. In order to ensure absolute stability of this method while retaining its accuracy and economy, it is suggested that the method of straight lines be used to extrapolate the solution from the previous time layer to the theoretical time layer. In a nine-node finite-difference scheme, it suffices to do this for the upper space layer, since the temperature distribution is already known on the lower layer.

Consequently, using extrapolation of the grid functions from the k-th layer to the (k + 1)-st layer by the method of straight lines, we obtain the following Cauchy problem for extrapolated temperatures

$$\frac{dT}{d\tau} = A\overline{T} + B; \quad (8)$$

$$\overline{T}(\tau^k) = T^k, \quad (9)$$

the solution of which is the expression

$$\overline{T} = T^k + (B - AT^k)\Delta\tau (E - A\Delta\tau/2) + O(\Delta\tau^3). \quad (10)$$

This solution does not include transformations of the matrix A, i.e., there is no decrease in the economy of the basic scheme. However, there is a substantial improvement in its stability, bringing the solution $\overline{T}$ closer to implicit grid functions T^{k+1} with the error $O(\Delta\tau^3)$.

In the case of an arbitrary boundary wl, the longitudinal coordinate lines r_j = const may enter and exit the theoretical region, which appreciably complicates approximation of the heat-flux balance (2) in the neighborhood of the boundary wl. To realize the approximation, we developed a method of inserting the specified theoretical region into a region with the classical boundary wl. Then the problem is solved continuously along the coordinate lines. Meanwhile, if the entire neighborhood of the node turns out to be outside the theoretical region, then the grid function takes a value of zero.

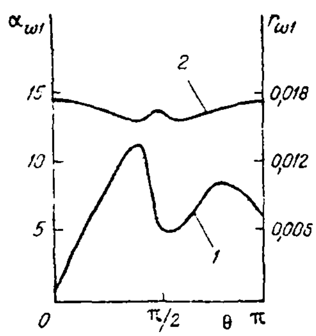

Fig. 3

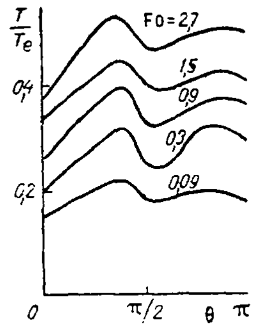

Fig. 4

Fig. 3. Heat-transfer coefficient α_{w1} (1), kW/(m²·K), and the coordinates of the external boundary $r_{w1} = f(\theta)$ (2), m.

Fig. 4. Change in the temperature field along the middle line $r = (R_0 + R_2)/2$.

In cases when the neighborhood of the boundary node is partly inside and partly outside the theoretical region, no energy is stored at the node if $r_{j-1} < r_{iw1} < r_{j-1/2}$. If $r_{j-1/2} < r_{iw1} < r_j$, the energy storage is considered in the volume $(r_{iw1} - r_{j-1/2})\Delta\theta$ and is included in the approximation of the balance (2). Here, the conservativeness of the finite-difference scheme is maintained.

In approximating the balance (2), we consider given heat fluxes by means of the unit functions:

$$\eta(x) = \begin{cases} 0, & \text{if } x \leqslant 0; \\ 1, & \text{if } x > 0, \end{cases}$$

which has made it possible to use the single balance approximation (2) for a wide range of different forms relative to the position of the boundary w1 and the nodes of the fixed grid.

Thus, balance approximation (2) has the form

$$\left(\frac{\alpha}{c_p}\right)_{iw1}(I_{e1}^{k+1} - c_{pw1}T_{iw1}^{k+1}) - \varepsilon_{w1}\sigma\,(T_{iw1}^{k+1})^4 - \bar{\Lambda}_n\left[-\frac{\Delta T}{\Delta r}\bigg|_{iw1}^{k+1} + \frac{\Delta \bar{T}}{r^{\nu}\Delta\theta}\bigg|_{iw1} f'\right] =$$
$$= \eta(r_{iw1} - r_{j-1/2})\,(\bar{c\rho})_{w1}(r_{iw1} - r_{j-1/2})\,\frac{\Delta T}{\Delta\tau}\bigg|_{iw1}\left[1 + \left(\frac{df}{r_{iw1}^{\nu}d\theta}\right)^2\right]^{-1/2}, \tag{11}$$

where $\bar{\Lambda}_n$ is found from Eq. (7).

To calculate the derivatives $df/r_{iw1}{}^{\nu}d\theta$ in (11) with a high degree of accuracy, the boundary w1 is approximated by cubic splines. As a result, the derivative $f' = dr_{w1}/r_{w1}{}^{\nu}d\theta$ represents a quadratic monomial in the variable θ.

In the next time layer in approximation (11), the longitudinal difference operators are represented implicitly, while the radial operators are calculated by means of extrapolation using the straight line method.

Figure 2 compares the solution obtained by the given method to the analytical solution in [4] in an anisotropic half-space

$$T = T_0\int_0^{\tau}(y/\tau'^{3/2})\,(\pi\rho c/4\lambda_{yy})^{1/2}\exp(-y^2\rho c/4\lambda_{yy}\tau')\,\times$$
$$\times\left\{\operatorname{erf}\left[\left(L + \frac{\lambda_{xy}}{\lambda_{yy}}y - x\right)\frac{1}{2}\left(\frac{\lambda_{yy}\rho c}{\lambda_\xi\lambda_\eta\tau'}\right)^{1/2}\right] + \operatorname{erf}\left[\left(L - \frac{\lambda_{xy}}{\lambda_{yy}}y + x\right)\frac{1}{2}\left(\frac{\lambda_{yy}\rho c}{\lambda_\xi\lambda_\eta\tau'}\right)^{1/2}\right]\right\}d\tau'$$

with the first boundary condition $T(x, 0, \tau) = \begin{cases} T_0, & |x| \leqslant L \\ 0, & |x| > L \end{cases}$ and a zero initial condition. Here, x and y are cartesian coordinates. The following values were taken as the input data: $\lambda_\xi = 4.2\cdot10^{-3}$ kW/(m·K); $\lambda_\eta = 4.2\cdot10^{-4}$ kW/(m·K); $\psi = 15°$; L = 0.02 m; T_0 = 1000 K; $c\rho$ = 600 kJ/(m³·K).

Comparison of the results shows that the method is satisfactorily accurate (error no worse than 5% for small numbers Fo). An increase in Fo is accompanied by a reduction in the error of the method.

Calculations were performed for the conditions in Fig. 3 to establish two-dimensional nonsteady temperature fields in anisotropic bodies with an arbitrarily assigned boundary.

Figure 4 shows the results of calculations, performed for the middle line $r = (R_0 - R_2)/2$, in the form of the relations $T/T_e = \varphi(\theta, Fo)$. It is evident from the figure that the temperature field is two-dimensional and quite unsteady.

We took the following initial data for the calculations: $T(r, \theta, 0) = 300$ K = const; $c\rho = 1500$ kJ/(m^3·K); $\lambda_\xi = 0.084$ kW/(m·K); $\lambda_\eta = 0.168$ kW/(m·K); $\psi = 30°$; $R_0 = 0.018$ m; $R_2 = 0.01$ m; $T_e = 2200$ K.

NOTATION

r, θ, curvilinear coordinates, m, rad; τ, time, sec; I, enthalpy, kJ/kg; T, temperature, K; α, heat-transfer coefficient, kW/(m^2·K); c, heat capacity, kJ/(kg·K); ρ, density, kg/m^3; ε, emissivity; σ, Stefan–Boltzmann constant; λ_ξ, λ_η, components of the principal thermal conductivity gensor, kW/(m·K); λ_{rr}, $\lambda_{r\theta}$, $\lambda_{\theta\theta}$, components of the thermal conductivity tensor, kW/(m·K); ψ, angle between the axis ξ and axis $\theta = 0$; Δr, $\Delta\theta$, $\Delta\tau$, space and time steps in numerical integration. Indices: w1, external boundary; w2, internal boundary; e1, gas on the external boundary; e2, gas on the internal boundary; ξ, η, principal axes of thermal conductivity; n, direction of the normal; k, preceding time layer; k + 1, theoretical time layer; i, j, node with the coordinates $\theta = \theta_i$, $r = r_j$.

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